Interaction between Kirchhoff vortices and point vortices in an ideal fluid

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Abstract. We consider the interaction of two vortex patches (elliptic Kirchhoff vortices) which move in an unbounded volume of an ideal incompressible fluid. A moment second-order model is used to describe the interaction. The case of integrability of a Kirchhoff vortex and a point vortex by the variable separation method is qualitatively analyzed. A new case of integrability of two Kirchhoff vortices is found. A reduced form of equations for two Kirchhoff vortices is proposed and used to analyze their regular and chaotic behavior.

1 Introduction.

The simplest example of planar vortex motion of an ideal fluid, other than that described by the point vortex model, was suggested by Kirchhoff [1]. He established that an elliptical vortex patch with semiaxes of a, b and a uniform vorticity ω inside uniformly rotates around its center with an angular velocity of $\Omega = (\omega\lambda)/(1+\lambda)^2$, $\lambda = a/b$. In this rotation, the fluid particles are involved in the absolute motion with a double angular velocity (Lamb, 1932). Lord Kelvin (1880) and Love (1893) showed that the Kirchhoff vortex is neutrally stable if and only if $a/b < 3$.

S. A. Chaplygin in 1899 generalized Kirchhoff's solution by introducing a uniform background vorticity into the unbounded fluid surrounding the elliptic vortex (this is the so-called Couette simple shearing motion). He established that the vortex will rotate with a certain angular velocity and change its contour (pulsate) in accordance with a certain law, which he obtained by integrating a system of two nonlinear differential equations. He also analyzed in detail the behavior of pressure in the fluid as a function of time.

Kida [2] and Neu [3] generalized Chaplygin's solution while being unaware of his work [4]. Chaplygin's works in many fields of mechanics are poorly known outside the Russian speaking world. Forgotten Chaplygin's works on two-dimensional vortex structures are discussed in the recent historical review by Meleshko and van Heijst [5]. A superposition of the solutions obtained by Kida and Neu is presented in the book by Newton[6]. It was found that the

dynamics of a Kirchhoff vortex can be reduced to a one-degree-of-freedom Hamiltonian system if the velocity of the external flow can be described by

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3, \\ \mathbf{u}_1 &= (\gamma_1 x, -\gamma_2 y, 0), \\ \mathbf{u}_2 &= (0, 0, \gamma_3 z), \\ \mathbf{u}_3 &= (-\gamma_4 y, \gamma_4 x, 0) \end{aligned} \tag{1}$$

under the condition $\gamma_1 - \gamma_2 + \gamma_3 = 0$ derived from the incompressibility condition, and. It is assumed that the elliptic patch is the section of the elliptical cylinder determined by the plane xy. Equation (1) yields Chaplygin's solution when $\gamma_1 = \gamma_2 = \gamma_4 \neq 0$, $\gamma_3 = 0$; Kida's solution when $\gamma_3 = 0$, $\gamma_1 = \gamma_2$; and Neu's solution when $\gamma_4 = 0$. Note that $\mathbf{u_1}$ corresponds to the uniform deformation field (which is induced at a distance by one point vortex [7], [8]); the field $\mathbf{u_2}$ corresponds to the extension along z axis, and the field $\mathbf{u_3}$ corresponds to uniform background rotation. The Hamiltonian system that describes this case has the form

$$\dot{\lambda} = -\frac{\partial H}{\partial \theta}, \quad \dot{\theta} = \frac{\partial H}{\partial \lambda}, \quad H(\lambda, \theta) = \ln \frac{(1+\lambda)^2}{\lambda} - \frac{1}{2} \frac{\gamma_1 + \gamma_2}{\omega(t)} \Big(\lambda - \frac{1}{\lambda} \Big) \sin 2\theta, \tag{2}$$

where the dot denotes differentiation with respect to the new time $\tau = \omega(t)(\lambda^2/(\lambda^2 - 1))t$, while $\omega(t) = \omega_0 e^{(\gamma_2 - \gamma_1)t}$, and ω_0 is the initial vorticity in the Kirchhoff vortex. If $\gamma_1 = \gamma_2$, the system (2) is autonomous and can be explicitly integrated. Qualitative analysis of this system is given in [2], [3], [7], [8].

When the coefficients in (1) are functions of time (for example, periodical), we obtain (at $\gamma_1 = \gamma_2$) a Hamiltonian system with one and a half

degrees of freedom with periodic perturbations, which was studied from the viewpoint of splitting of separatrices and the appearance of stochastic behavior [9], [10]. The problem of advection of a passive particle of the fluid in the Kida flow was discussed in [11]. Examination of Poincaré sections allows us to conclude that the advection is chaotic, i. e., Lagrangian turbulence exists. The stability of the elliptic vortex in a Kida flow (uniform deformation) was studied in Ref. [12].

Finally, we will mention the generalized Kirchhoff solution obtained by Polvani and Flierl [13] and corresponding to a family of embedded confocal elliptic vortices with the appropriate distribution of vorticity. In that work, the stability conditions, which generalize the Kelvin–Love conditions, were obtained for a system of two confocal elliptic vortices.

2 Moment model of Kirchhoff vortex interaction (vortex patch dynamics).

The second-order moment model [14] represents a higher level of approximation in the description of hydrodynamic vorticity as compared with the point vortex model. This model considers elliptic patches with a specified vorticity, which move in a two-dimensional incompressible fluid. This theory was suggested by Melander, Zabusky, and Stychek (MZS model) in Ref. [14]. Rotation of two Kirchhoff vortices in the presence of central symmetry is analyzed in [15]. The nonsymmetrical situation of interaction between two vortex patches is discussed in [16], where the same moment model, which

describes merging of vortices, is compared with a pseudospectral model (in which Euler equations with a weak dissipation are solved).

Let us describe this model in more detail. The second-order moment model, describing Kirchhoff vortex interactions, is derived from two basic assumptions:

1. the distance between vortices in the process of evolution is much greater than the vortex size, therefore the vortices will maintain their elliptic shape;

2. the third and higher moments in the expansion of the Hamiltonian can be neglected.

Under these assumptions, the equations of motion of elliptic vortices can be written in the Hamiltonian form with nonlinear Poisson brackets [14]

$$\begin{gathered} \dot{x}_k = \{x_k, H\}, \quad \dot{y}_k = \{y_k, H\}, \quad \dot{\varphi}_k = \{\varphi_k, H\}, \quad \dot{\lambda}_k = \{\lambda_k, H\}, \\ \{x_i, y_j\} = \frac{1}{\Gamma_i}\delta_{ij}, \quad \{\varphi_i, \lambda_j\} = \frac{8\pi}{\Gamma_i S_i}\frac{\lambda_i^2}{1-\lambda_i^2}\delta_{ij}, \end{gathered} \tag{3}$$

and the Hamiltonian is

$$\begin{gathered} H = H_1 + H_2 + H_3, \\ H_1 = -\frac{1}{8\pi}\sum_{k=1}^{N}\Gamma_k^2 \ln\frac{(1+\lambda_k)^2}{4\lambda_k}, \quad H_2 = -\frac{1}{8\pi}{\sum_{k,p}^{N}}' \Gamma_k\Gamma_p \ln M_{kp}, \\ H_3 = -\frac{1}{32\pi^2}{\sum_{k,p}^{N}}' \frac{\Gamma_k\Gamma_p}{M_{kp}}\left(S_p\frac{1-\lambda_p^2}{\lambda_p}\cos(2(\theta_{kp}-\varphi_p)) \right. \\ \left. + S_k\frac{1-\lambda_k^2}{\lambda_k}\cos(2(\theta_{kp}-\varphi_k))\right), \end{gathered} \tag{4}$$

where Γ_k, S_k are the total intensity and the area of the elliptic vortex with index k; M_{kp} is the squared distance between the centers of the k-th and p-th

vortices ($M_{kp} = (x_k - x_p)^2 + (y_k - y_p)^2$); φ_k is the slope of the k-th ellipse with respect to the x axis; and θ_{kp} is the angle between the x-axis and the straight line connecting the centers of the k-th and p-th ellipses (see Fig. 1).

The area of each ellipse remains constant (S_k = const) by virtue of Kelvin's theorem on the conservation of circulation in an ideal medium [17], [18]; therefore, these areas are parameters of the model considered.

The components of the Hamiltonian have the following physical meaning:

H_1 describes the action of an elliptic vortex on itself, see (2);

H_2 describes the interaction between equivalent point vortices;

H_3 describes the interaction between vortices with second-order moments taken into account.

In addition to the Hamiltonian H, equations (3) have a *noncommutative* set of first integrals

$$Q = \sum_k^N \Gamma_k x_k, \ P = \sum_k^N \Gamma_k y_k, \ I = \sum_k^N \Gamma_k \left[x_k^2 + y_k^2 + \frac{S_k}{4\pi} \frac{1 + \lambda_k^2}{\lambda_k} \right], \quad (5)$$

which correspond to the translational and rotational invariance of the system in the absolute space.

Integrals Q, P, I satisfy the following commutation relations:

$$\{Q, P\} = \sum_{i=1}^N \gamma_i, \quad \{P, I\} = -2Q, \quad \{Q, I\} = 2P. \quad (6)$$

Therefore these relations are not sufficient even for the integrability of a system of two Kirchhoff vortices; this system will be further reduced to a system with two degrees of freedom. However, the problem of the dynamics of a Kirchhoff vortex and a point vortex (a system with three degrees of freedom) is integrable.

A more general model of interaction between elliptic vortex patches was suggested by D. Dritschel and B. Legras [19], [20]. In some sense, this is an intermediate model between the MZS-model and an exact description determined by the contour dynamics method (in the contour dynamics method, the elliptic shape of the patch does not persist, while the model [19], [20] is derived with no allowance made for the velocity field responsible for the nonelliptic part of the interaction). However, this model is more complex and in many cases it is sufficient to use the MZS-model.

3 Interaction of a Kirchhoff vortex with N point vortices. Integrable case for $N = 1$.

Let us denote the coordinates of the center of the elliptic vortex by (x_0, y_0) and those of the point vortices, by (x_i, y_i), $i = 1, \ldots, N$; the equations describing the dynamics of this system also can be written in Hamiltonian form with a Poisson bracket and the Hamiltonian in the form

$$\{x_i, y_i\} = \Gamma_i^{-1}\delta_{ij}, \quad i, j = 0, 1, \ldots, N, \quad \{\varphi, \lambda\} = \frac{8\pi}{\Gamma_0 S}\frac{\lambda^2}{1-\lambda^2},$$

$$H = H_1 + H_2 + H_3,$$

$$H_1 = -\frac{1}{8\pi}\Gamma_0^2 \ln\frac{(1+\lambda)^2}{4\lambda}, \quad H_2 = -\frac{1}{8\pi}\sum_{k,p=0}^{N}{}' \Gamma_k\Gamma_p \ln M_{kp}, \tag{7}$$

$$H_3 = -\frac{\Gamma_0 S}{16\pi^2}\sum_{k=1}^{N}\frac{\Gamma_k}{M_{k0}}\frac{1-\lambda^2}{\lambda}\cos 2(\theta_k - \varphi),$$

where Γ_0 is the intensity of the Kirchhoff vortex with the semiaxes ratio of λ with the angle φ, determining its orientation (see Fig. 1), S is the

area of the ellipse, θ_k is the angle between the x-axis and the straight line connecting the center of the Kirchhoff vortex and the k-th point vortex, and $M_{kp} = (x_k - x_p)^2 + (y_k - y_p)^2$. Hereafter, we will assume without loss of generality that the intensity of the Kirchhoff vortex is positive ($\Gamma_0 > 0$).

The integrals of motion corresponding to the group $E(2)$ of motions of the plane are determined by relationships (5), where it should be assumed that $S_0 = S$, and $S_i = 0$, $i = 1, \ldots, N$; their commutation relationships are analogous to (6). As a consequence of the existence of integrals, we obtain [21]:

Proposition 1 *The system of an interacting Kirchhoff vortex and one point vortex* ($N = 1$) *is completely integrable.*

This was first shown by Lebedev [21] and somewhat later was established independently by Riccardi and Piva [22]. Here we will give a geometric analysis of the motion of vortices to improve the results of [21].

For explicit integration and qualitative analysis, we perform a reduction to one degree of freedom. Let us consider new (relative) variables

$$\psi = 2(\theta - \varphi), \quad \rho = \frac{1}{2}c\Big(\lambda + \frac{1}{\lambda}\Big), \quad z = M_{10} = (x_1 - x_0)^2 + (y_1 - y_0)^2, \tag{8}$$

where $c = \Gamma_0 S/8\pi$. All these functions (8) commute with the integrals (5), i. e., they are invariants of the group of motions of the plane $E(2)$ and are closed with respect to the Poisson bracket

$$\{\psi, \rho\} = 1, \quad \{\psi, z\} = -4(\Gamma_0^{-1} + \Gamma_1^{-1}), \quad \{\rho, z\} = 0. \tag{9}$$

The Poisson structure (9) has a linear Kazimir function

$$D = \Gamma_0 z + 4(1 + \alpha)\rho, \quad \alpha = \Gamma_0^{-1}\Gamma_1. \tag{10}$$

Eliminating z with the use of equation (10), we obtain the Hamiltonian of the system (up to the constant) with one degree of freedom:

$$H_{\mp} = \frac{\Gamma_0^2}{8\pi}\left(-\ln(c+\rho) - 2\alpha\ln(D - 4(1+\alpha)\rho) \mp 8\alpha\frac{\sqrt{\rho^2 - c^2}}{D - 4(1+\alpha)\rho}\cos\psi\right), \tag{11}$$

where, according to (9), the variables ψ, ρ are canonical.

The different signs in the Hamiltonian (11) appear due to the nonuniqueness of the inverse transformation (8) for $\rho(\lambda)$, which has the form

$$\lambda = \begin{cases} \dfrac{\rho - \sqrt{\rho^2 - c^2}}{c}, & 0 < \lambda \leq 1 \\ \dfrac{\rho + \sqrt{\rho^2 - c^2}}{c}, & \lambda > 1 \end{cases} \tag{12}$$

(the upper sign in (11) corresponds to the case $\lambda < 1$).

The reduction by (to two degrees of freedom) for arbitrary N can be made in a similar manner. The new variables for the reduced system can be chosen as follows:

$$\psi_k = 2(\theta_k - \varphi), \quad \rho = \frac{1}{2}c(\lambda + \lambda^{-1}),$$
$$M_{ik} = (x_i - x_k)^2 + (y_i - y_k)^2, \quad i = 0, \dots, N, \quad k = 1, \dots, N.$$

Qualitative analysis of the relative motion for $N = 1$. According to (8), the domains of the variables ψ, ρ are determined by the inequalities

$$0 \leq \psi < 4\pi, \quad c \leq \rho.$$

Each point of this half-strip corresponds to a pair of possible mutual arrangements of the elliptic and point vortices (12), corresponding to $\lambda < 1$ and $1 < \lambda$ (see Fig. 2).

It can be shown that the *relation*

$$\rho = c, \quad i.\,e.\ \lambda = 1,$$

corresponds to the case when elliptic vortex becomes circular (Rankin vortex).

The trajectories of the reduced system are determined by the contour lines of the Hamiltonian (11); therefore, the trajectories of the system in domains $\lambda < 1$ and $\lambda > 1$ can be obtained with the help of the substitution

$$\psi \to \psi + \pi, \quad \rho \to \rho.$$

Moreover, the Hamiltonian (11) is 2π-periodic with respect to ψ. Therefore, we will restrict ourselves to the analysis of the contour lines of the Hamiltonian H_- in the domain

$$0 \le \psi < 2\pi, \quad c \le \rho.$$

Normalizing the variable and the integral

$$\rho = c\widetilde{y}, \quad D = 4c|1+\alpha|\widetilde{D} \tag{13}$$

and eliminating "insignificant" constants in (11), we find that the trajectories of the system are determined by the level curves of the function

$$\begin{aligned} \widetilde{H}_- &= -\ln(1+\widetilde{y}) - 2\alpha\ln(\widetilde{D}-\widetilde{y}) - \frac{2\alpha}{1+\alpha}\frac{\sqrt{\widetilde{y}^2-1}}{\widetilde{D}-\widetilde{y}}\cos\psi, \quad 1+\alpha > 0, \\ \widetilde{H}_- &= -\ln(1+\widetilde{y}) - 2\alpha\ln(\widetilde{D}+\widetilde{y}) - \frac{2\alpha}{|1+\alpha|}\frac{\sqrt{\widetilde{y}^2-1}}{\widetilde{D}+\widetilde{y}}\cos\psi, \quad 1+\alpha < 0. \end{aligned} \tag{14}$$

According to [14], the equations describing the dynamics of a Kirchhoff vortex interacting with a point vortex are valid only at a sufficiently large

distance from the Kirchhoff vortex. Without specifying the exact domain where the results thus obtained are applicable, we give here a trajectory of the reduced system (11) and mark the domain occupied by the Kirchhoff vortex. In accordance with (10), (13), the elliptic vortex in the plane ψ, $\widetilde{y}$ occupies the domain determined by the relationships

$$\begin{aligned} &\widetilde{D}-\widetilde{y} \leq \frac{2}{1+\alpha}(\widetilde{y}-\sqrt{\widetilde{y}^2-1}\cos\psi), \quad 1+\alpha>0, \\ &\widetilde{D}+\widetilde{y} \leq \frac{2}{|1+\alpha|}(\widetilde{y}-\sqrt{\widetilde{y}^2-1}\cos\psi), \quad 1+\alpha<0. \end{aligned} \tag{15}$$

The domain occupied by the Kirchhoff vortex is shaded in the figures below.

Stability of the circular vortex. Let us make the canonical change of variables

$$\widetilde{y} = 1+\frac{u^2+v^2}{2}, \quad \psi = \arctan\frac{u}{v}. \tag{16}$$

Now the Hamiltonian (14) near $\widetilde{y}=1$ can be written as

$$\begin{aligned} \widetilde{H} &= \text{const} - \frac{2\sqrt{2}\alpha}{(1+\alpha)(\widetilde{D}-1)}v - \frac{1}{2}\frac{\widetilde{D}-1-4\alpha}{\widetilde{D}-1}(u^2+v^2)+\dots, \quad 1+\alpha>0; \\ \widetilde{H} &= \text{const} + \frac{2\sqrt{2}\alpha}{(1+\alpha)(\widetilde{D}+1)}v - \frac{1}{2}\frac{\widetilde{D}+1+4\alpha}{\widetilde{D}+1}(u^2+v^2)+\dots, \quad 1+\alpha<0. \end{aligned}$$

Thus, after the identification of (16), we obtain that the origin of coordinates $u=v=0$ is not a fixed point, i. e., in the presence of point vortex, the circular vortex is locally unstable with respect to elliptic deformations.

Let us qualitatively describe the structure of the phase portrait on the ψ, $\widetilde{y}$ plane for different values of the parameters α, $\widetilde{D}$. As follows from (10) and (14), the phase space of the reduced system and the respective phase portraits change depending on the sign of $(1+\alpha)$. Let us consider each case separately.

$1+\alpha>0$ (Fig. 3). In this case, the phase space is the rectangle on the plane ψ, $\widetilde{y}$:

$$0 \le \psi < 2\pi, \quad 1 \le \widetilde{y} < \widetilde{D}. \tag{17}$$

The opposite sides of this rectangle $\psi = 0$ and $\psi = 2\pi$ are identified with one another. The segment $\widetilde{y} = 1$ corresponds to the case of a circular Kirchhoff vortex, and $\widetilde{y} = \widetilde{D}$ is the case when the point vortex is in the center of the elliptic vortex. In both cases the definition of the angle ψ makes no sense (i. e., this is a singularity similar to the origin for polar coordinates). In this case, the phase space can be identified with a two-dimensional sphere.

Remark 1 The identification for the sphere $\xi_1^2+\xi_2^2+\xi_3^2 = 1$ embedded into $\mathbb{R}^3$ can be explicitly specified, for example, in the following manner

$$\xi_3 = \frac{2\widetilde{y} - \widetilde{D} - 1}{\widetilde{D} - 1}, \quad \xi_1 = \sqrt{1-\xi_3^2}\cos\psi, \quad \xi_2 = \sqrt{1-\xi_3^2}\sin\psi. \tag{18}$$

Fig. 3 gives a characteristic appearance of phase portraits corresponding to the three different parameter domains. The appearance of the portrait is completely determined by the critical points of the Hamilton function (14), which, as follows from (14), lie on the straight lines $\psi = 0$, $\psi = \pi$. In this case the point vortex lies on the principal axes of the elliptic Kirchhoff vortex.

Each straight line $\psi = 0$ and $\psi = \pi$ (in the nondegenerate case) contains either 0 or 2 critical points of the function (14).

The domain (17) can contain either 2 or 0 (nondegenerate) critical points of the function (14) that do not lie on the segments $\widetilde{y} = 1$, $\widetilde{y} = \widetilde{D}$. Both critical points lie either on the straight line $x = 0$ or on the straight line $x = \pi$. The critical point nearest to the segment $\widetilde{y} = 1$ always corresponds

to a stable fixed point of the reduced system.

A criterion for the global stability of the Kirchhoff vortex at $1+\alpha>0$ can be formulated within the framework of the model considered:

if the Hamilton function has critical points in the domain (17), the perturbations of the Rankin vortex remain constrained.

Indeed, in this case, as can be seen from Fig. 3, there always exists an invariant curve that limits the extent of the Kirchhoff vortex deformation and inhibits its merging with the point vortex.

$\mathbf{1}+\alpha<\mathbf{0}$ **(Fig. 4).** In this case, the domain of motion in the ψ, $\widetilde{y}$-plane is noncompact:

$$0 \le \psi < 2\pi, \quad \widetilde{y} \ge \max\left(1, -\widetilde{D}\right) = \widetilde{y}_0.$$

Carrying out the (canonical) change of coordinates

$$\widetilde{y} = \widetilde{y}_0 + \frac{u^2+v^2}{2}, \quad \psi = \arctan\frac{u}{v},$$

we find that the phase space is the (u,v)-plane and the straight line $\widetilde{y}=\widetilde{y}_0$ corresponds to the origin of coordinates. The structure of the phase portrait (the shape of the domain occupied by the Kirchhoff vortex) depends on the sign of $\widetilde{D}$.

When $\widetilde{D}>0$, there is a stable periodic solution on the straight line $\psi=\pi$ (Fig. 4a). For $\widetilde{D}<0$, an unstable periodic solution exists at $\psi=0$ (Fig. 4b).

$1+\alpha=0$ **(the case of a vortex pair).** The transformation (13) cannot be made in this case, therefore we carry out the normalization

$$\rho = c\widetilde{y}, \quad D = 4c\widetilde{D} = \Gamma_0 M_{10} = \text{const}. \tag{19}$$

Thus, in this case, the distance between vortices remains constant $(M_{10} = \text{const})$, and changes take place only in the mutual arrangement of the vortices. The variables ψ, $\widetilde{y}$ are determined only in the half-strip

$$0 \leq \psi \leq 2\pi, \quad 1 < \widetilde{y}.$$

In this case the trajectories are determined by the contour lines of the Hamiltonian H_{-} in (11), which, after the elimination of constant values can be written as

$$\widetilde{H}_{-} = -\ln(1+\widetilde{y}) + \frac{2}{\widetilde{D}}\sqrt{\widetilde{y}^2 - 1}\cos\psi, \tag{20}$$

The domain occupied by the elliptic vortex on the domain of the variables is determined by the inequality

$$\widetilde{D} \geq 2(\widetilde{y} - \sqrt{\widetilde{y}^2 - 1}\cos\psi). \tag{21}$$

Whence it follows that for $\widetilde{D} < 0$ the entire plane $(\psi, \widetilde{y})$ is occupied by the elliptic vortex; therefore, we assume that $\widetilde{D} > 0$.

For $\widetilde{D} < \widetilde{D}_* = \sqrt{22 + 10\sqrt{5}}$, the phase portrait contains no periodic solutions (Fig. 5a).

When $\widetilde{D} > \widetilde{D}_*$, two periodic solutions appear in the phase portrait on the axis $\psi = 0$; one of these solutions is stable, while the other is unstable (Fig. 5b,c)

4 Interaction between two Kirchhoff vortices

Reduction to a system with two degrees of freedom. The dynamics of two Kirchhoff vortices can be described by the Hamiltonian (4),

which can be represented in the form

$$\begin{gathered}
H = H_1 + H_2 + H_3, \\
H_1 = -\frac{\Gamma_1^2}{8\pi} \ln \frac{(1+\lambda_1)^2}{4\lambda_1} - \frac{\Gamma_2^2}{8\pi} \ln \frac{(1+\lambda_2)^2}{4\lambda_2}, \\
H_2 = -\frac{\Gamma_1 \Gamma_2}{4\pi} \ln M, \\
H_3 = -\frac{\Gamma_1 \Gamma_2}{16\pi^2 M} \left(\frac{S_1(1-\lambda_1^2)}{\lambda_1} \cos 2(\theta - \varphi_1) - \frac{S_2(1-\lambda_2^2)}{\lambda_2} \cos 2(\theta - \varphi_2) \right), \\
M = (x_1 - x_2)^2 + (y_1 - y_2)^2, \quad \theta = \theta_{12} = \pi + \theta_{21} = \arctan \frac{y_2 - y_1}{x_2 - x_1}.
\end{gathered} \tag{22}$$

The particular solution of the system (22), at which $\gamma_1 = \gamma_2$ and the vortices are centrally symmetrical with respect to one another is suggested and studied by Melander *et al.* [15]. In that paper, the conditions for merging of two vortex patches are given. These conditions, within the framework of the moment model, were found to be equivalent to the collapse of two Kirchhoff vortices, during which their centers coincide after a finite time.

Let us consider the system of relative variables

$$\begin{gathered}
\psi_i = 2(\theta - \varphi_i), \quad \rho_i = \frac{c_i}{2} \left(\lambda_i + \frac{1}{\lambda_i} \right), \quad c_i = \frac{\Gamma_i S_i}{8\pi}, \quad i = 1, 2. \\
z = \frac{1}{4} M.
\end{gathered} \tag{23}$$

These variables commute with the integrals (5); they are closed with respect to the Poisson brackets (3), and their commutation relationships have the form

$$\{\psi_i, \rho_i\} = \delta_{ij},\ \{\psi_i, z\} = -(\Gamma_1^{-1} + \Gamma_2^{-1}),\ \{\rho_i, z\} = 0,\ i, j = 1, 2. \tag{24}$$

The Poisson structure (24) has a linear Kazimir function (integral of

motion)

$$D = z + (\Gamma_1^{-1} + \Gamma_2^{-1})(\rho_1 + \rho_2). \tag{25}$$

As follows from (24), (25), in the case of the vortex pair ($\Gamma_1 + \Gamma_2 = 0$), the distances between the vortex centers remain constant.

Eliminating z using the integral (25) and expressing the Hamiltonian (22) in terms of the variables ψ_1, ρ_1, we obtain the Hamiltonian of the reduced canonical system with two degrees of freedom

$$\begin{aligned} H = &-\frac{\Gamma_1^2}{8\pi}\ln(c_1+\rho_1) - \frac{\Gamma_2^2}{8\pi}\ln(c_2+\rho_2) \\ &-\frac{\Gamma_1\Gamma_2}{4\pi}\ln\big(D - (\Gamma_1^{-1}+\Gamma_2^{-1})(\rho_1+\rho_2)\big) \\ -\frac{\Gamma_1\Gamma_2}{4\pi(D-(\Gamma_1^{-1}+\Gamma_2^{-1})(\rho_1+\rho_2))}&\left(\pm\frac{\sqrt{\rho_1^2-c_1^2}}{\Gamma_1}\cos\psi_1 \mp \frac{\sqrt{\rho_2^2-c_2^2}}{\Gamma_2}\cos\psi_2\right), \end{aligned} \tag{26}$$

where the upper sign corresponds to the condition $\lambda_i > 1$, and the lower sign to the condition $0 < \lambda_i \leq 1$ (see. (12)).

Absolute motion. Let us find the equations determining the positions (x_1, y_1) and (x_2, y_2) and orientation φ_1, φ_2 of two Kirchhoff vortices in stationary space if their relative arrangement determined by the system (26) is assumed to be known: $\rho_i = \rho_i(t)$, $\psi_i = \psi(t)$, $i = 1, 2$. Direct calculations show that the slopes of the principal (major) semiaxes of the ellipse are determined by the quadratures

$$\dot{\varphi}_i = \frac{\Gamma_i c_i}{S_i}\frac{\rho_i + \sqrt{\rho_i^2 - c_i^2}}{\left(\rho_i + c_i + \sqrt{\rho_i^2-c_i^2}\right)^2} - \frac{\Gamma_1\Gamma_2}{2\pi\Gamma_i M}\frac{\left(\rho_i + \sqrt{\rho_i^2-c_i^2}\right)\rho_i\cos\psi_i}{\rho_i^2 - c_i^2 + \rho_i\sqrt{\rho_i^2-c_i^2}}. \tag{27}$$

If $\Gamma_1 + \Gamma_2 \neq 0$, the positions of the vortex centers can be found from the system of linear equations with coefficients explicitly depending on time:

$$
\begin{aligned}
x_1 &= \frac{Q + \Gamma_2 \Delta x}{\Gamma_1 + \Gamma_2}, \quad x_2 = \frac{Q - \Gamma_1 \Delta x}{\Gamma_1 + \Gamma_2}, \quad y_1 = \frac{P + \Gamma_2 \Delta y}{\Gamma_1 + \Gamma_2}, \quad y_2 = \frac{P - \Gamma_2 \Delta y}{\Gamma_1 + \Gamma_2}; \\
\Delta \dot{x} &= -\frac{\Gamma_1 + \Gamma_2}{\pi^2 M^2} \left(\frac{\pi M}{2} \Delta y + \frac{S_1 \sqrt{\rho_1^2 - c_1^2}}{4c_1} (\Delta x \sin \psi_1 + \Delta y \cos \psi_1) \right. \\
&\qquad \left. + \frac{S_2 \sqrt{\rho_2^2 - c_2^2}}{4c_2} (\Delta x \sin \psi_2 + \Delta y \cos \psi_2) \right); \\
\Delta \dot{y} &= \frac{\Gamma_1 + \Gamma_2}{\pi^2 M^2} \left(\frac{\pi M}{2} \Delta x + \frac{S_1 \sqrt{\rho_1^2 - c_1^2}}{4c_1} (\Delta x \cos \psi_1 - \Delta y \sin \psi_1) \right. \\
&\qquad \left. + \frac{S_2 \sqrt{\rho_2^2 - c_2^2}}{4c_2} (\Delta x \cos \psi_2 - \Delta y \sin \psi_2) \right)
\end{aligned}
\tag{28}
$$

where $\Delta x = x_1 - x_2$, $\Delta y = y_1 - y_2$, and Q, P are integrals (5).

Integrable case $\Gamma_1 + \Gamma_2 = 0$. In the case of the vortex pair $\Gamma_1 = -\Gamma_2$, the Hamiltonian (26) can be split into two independent Hamiltonians

$$H = H_1(\psi_1, \rho_1) + H_2(\psi_2, \rho_2),$$

and the system can be integrated by the method of separation of variables. Thus, we obtain a new nontrivial integrable case for the vortex dynamics. This case of integrability was first mentioned in our review [23]. As follows from equations (28), in this case

$$x_1 - x_2 = \text{const}, \quad y_1 - y_2 = \text{const},$$

and hence

$$\theta = \arctan \frac{y_2 - y_1}{x_2 - x_1} = \text{const}.$$

Thus, the length and orientation of the segment connecting the centers of vortices remain unchanged during the motion (see Fig. 1).

Poincaré sections and integrability. The system (26) is not integrable in the general case. This is demonstrated by the chaotic trajectories constructed at $\Gamma_1 = \Gamma_2$ with the use of a Poincaré section shown in Fig. 6. It should be mentioned that there are few returning trajectories in the case of interaction between Kirchhoff vortices, which hampers the numerical analysis. Fig. 7 gives numerically plotted separatrices of a hyperbolic fixed point of the Poincaré mapping. Their transversal intersection (established by computer analysis) is an indication that the problem of Kirchhoff vortex motion is nonintegrable. This fact was mentioned as a supposition in [14]. No analytical proof of the nonintegrability of two Kirchhoff vortices within the framework of the second-order moment model has been obtained yet.

5 Acknowledgements

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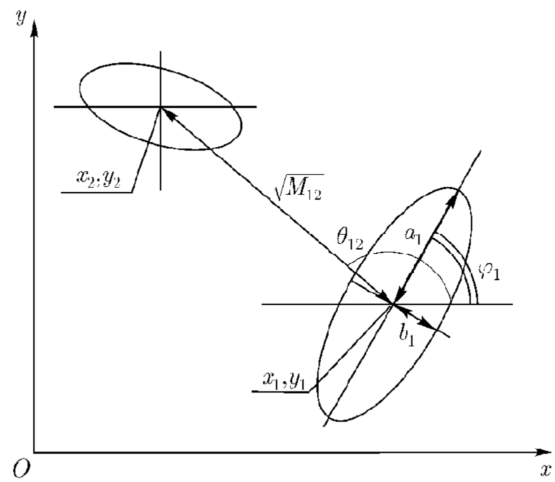


Figure 1: Elliptic vortex patches

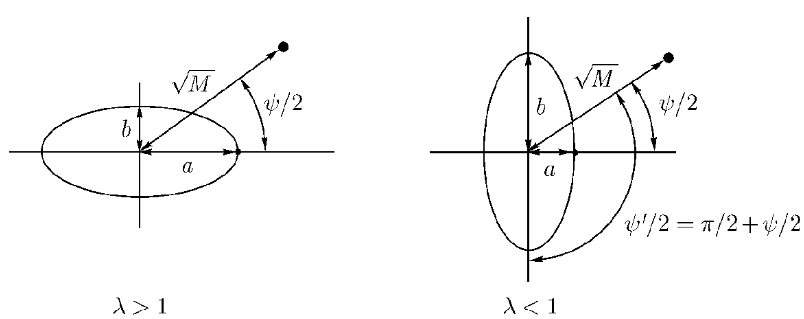


Figure 2: Mutual arrangement of the elliptic and point vortices with respect to a chosen point on the contour of the elliptic vortex in the case where $\lambda < 1$ and $\lambda > 1$.

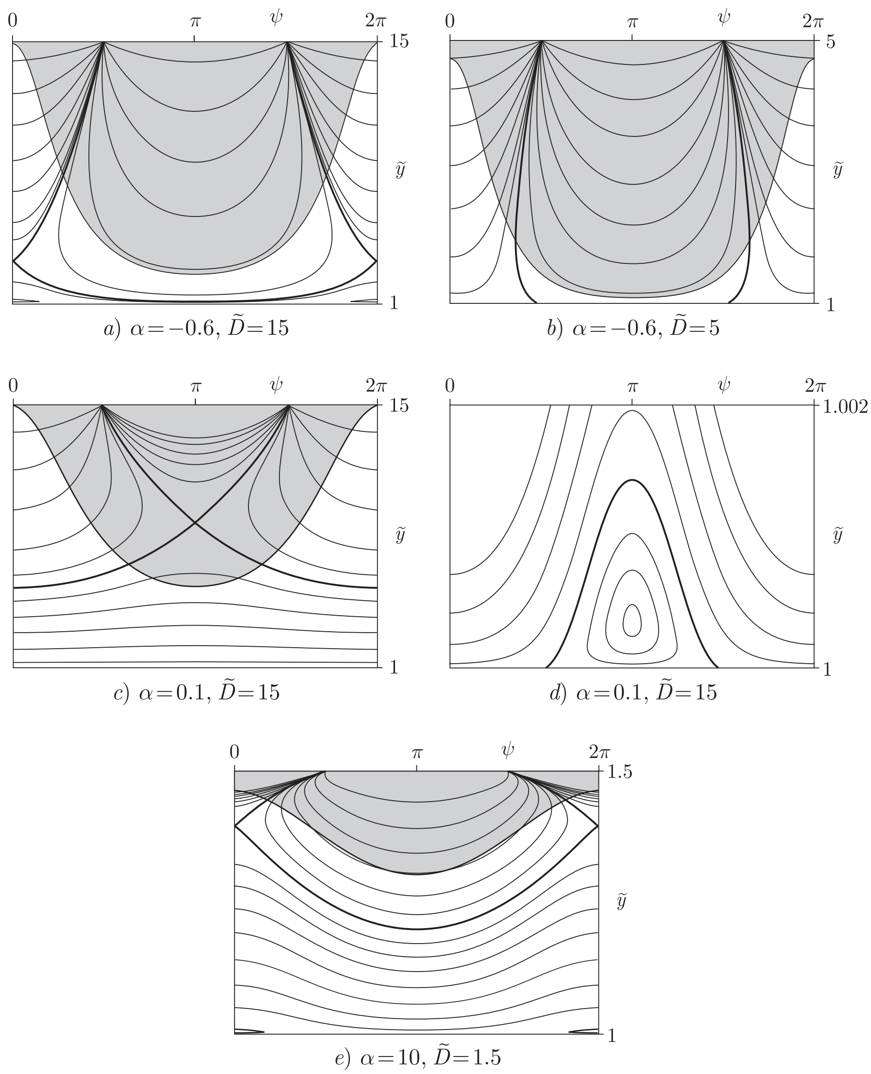
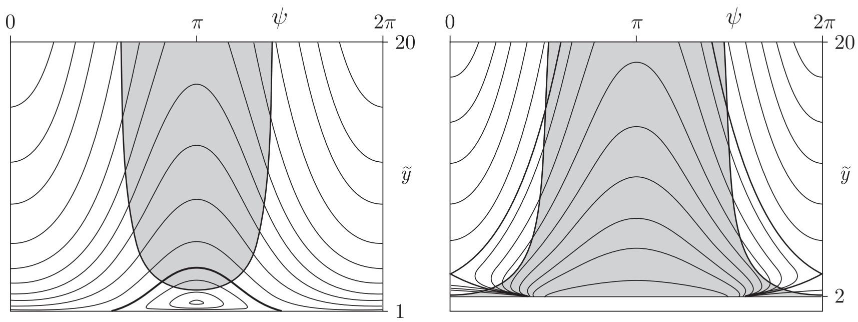


Figure 3: Phase portraits for $1+\alpha > 0$ (the domain occupied by the Kirchhoff vortex is colored gray). Figure 3d is an enlarged part of figure 3c near the lower segment.



a) $\alpha = -2.2$, $\widetilde{D} = 1.5$ *b)* $\alpha = -2.2$, $\widetilde{D} = -2$

Figure 4: Phase portraits for $1+\alpha < 0$ (the domain occupied by the Kirchhoff vortex is colored gray).

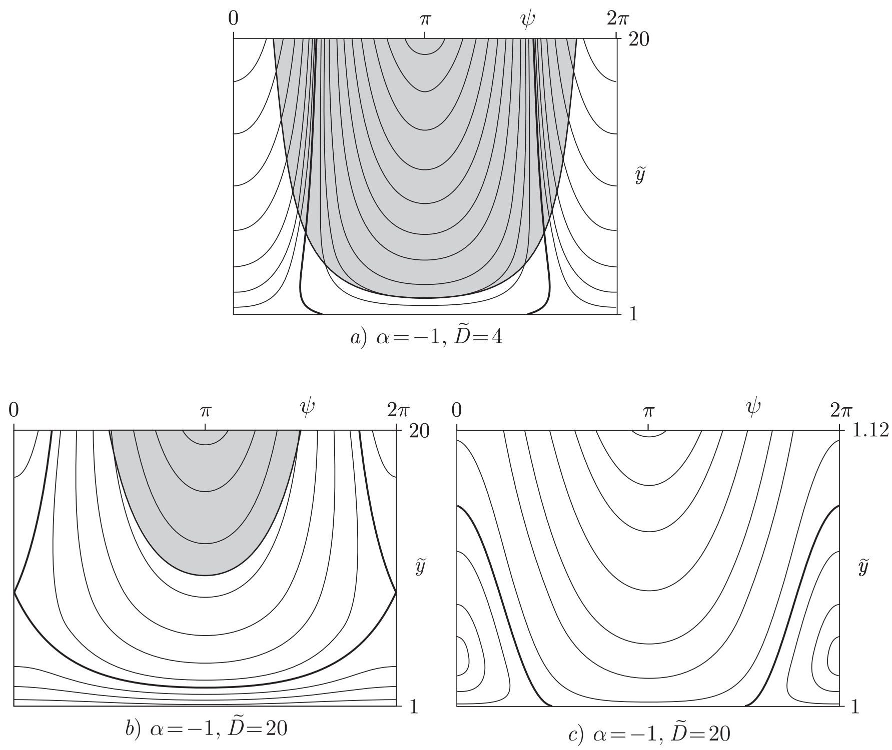


Figure 5: Phase portrait for $1+\alpha = 0$ (the domain occupied by the Kirchhoff vortex is colored gray). Figure 5*c* is an enlarged part of figure 5*b* near the lower segment.

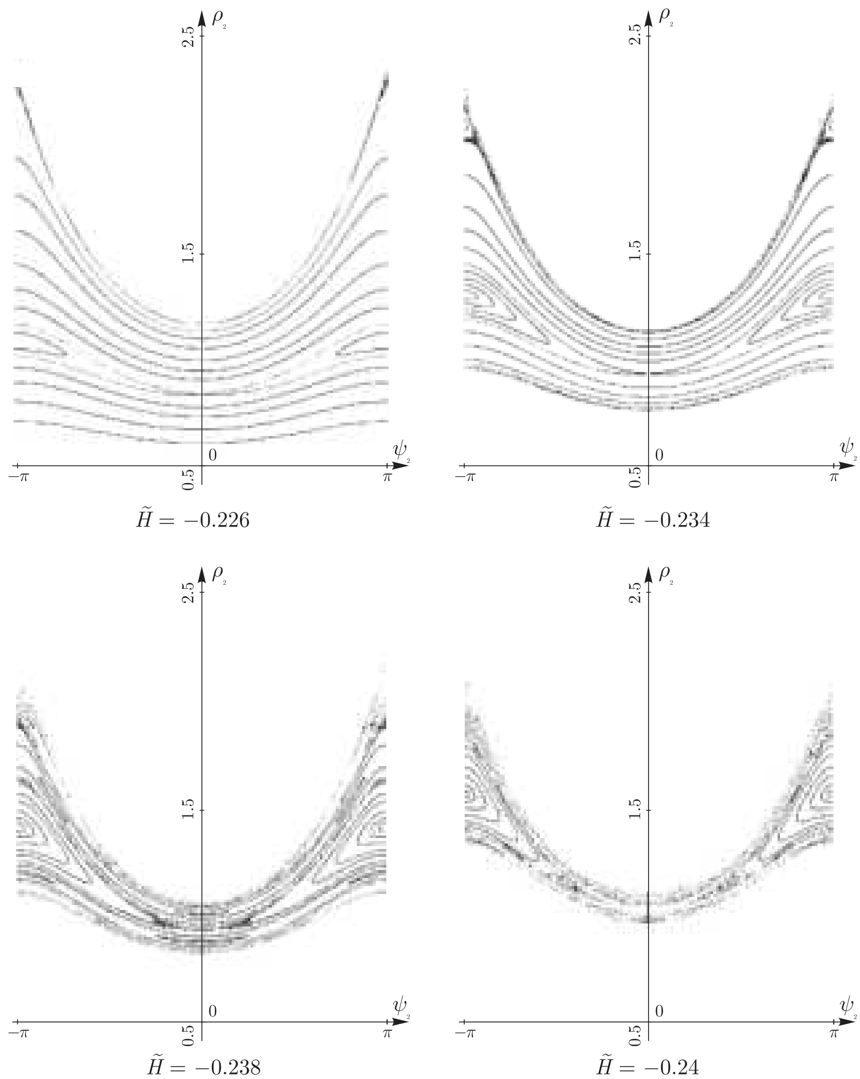


Figure 6: Poincaré mapping for the reduced system (26) in the problem of two elliptic vortices. The section by the plane $\psi_1 = 0$ was chosen for the following parameter values: $\gamma_1 = \gamma_2 = 1$, $S_1 = S_2 = 0.3$ and $D = 22$.

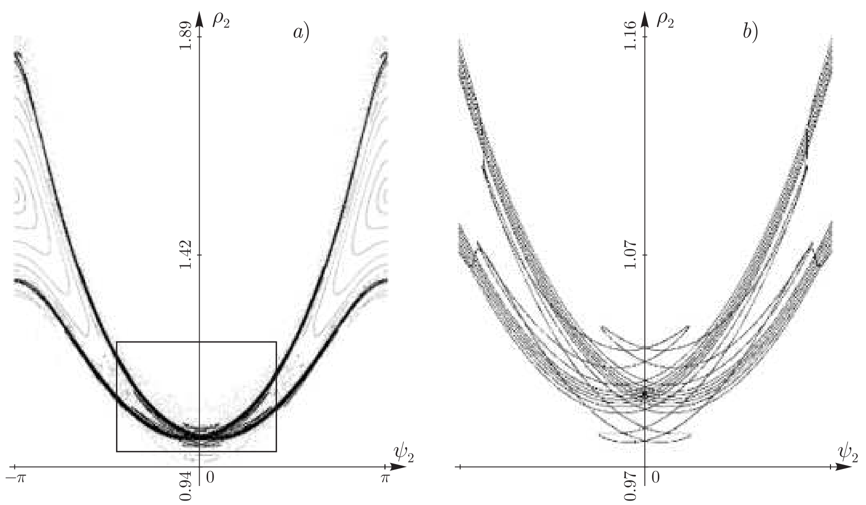


Figure 7: Separatrices for the last phase portrait shown in Fig. 6 ($\widetilde{H} = -0.24$). Fig. 7b is the enlarged central part of figure 7a.